Positive cones and Non-negativity Domains of Subsets of Groups

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Abstract:

a total and antisymmetric subset D of a subset of a group X is called a nonnegativity domain of A, the non-negativity domain of subsets of additive groups are investigated in [8]. In this paper we will compare the non-negative domain with the positive cone, which is a subset of a partially ordered group G, positive cone defined as $G^{+} = U(e) = \{x \in G: x \geq e\}$ of all positive (integral) elements of G. However, in order to make nonnegativity domains comparably with positive cones, we shall discuss nonnegativity domains for multiplicative groups. glavosits sza'z defined the non-negativity domains as a subset of the group, it is not necessary for nonnegativity domain to be total in the group itself which distinguishes it from what is called a positive cone.

This paper contains some basic material as preorder (or quasiorder) relations, definitions and properties of partially ordered sets, partially ordered relation, equivalent relation, lattice sets, convex sets and directed sets.

We also investigated ordered relations determined by subsets of groups having some spatial of properties.

المخروطات الموجبة والنطاقات غير السالبة على المجموعات الجزئية من الزمر

الملخص:

عرف (غلافوسيتس، سازا / glavosits, szaz) في سنة 2004 النطاقات غير السالبة لمجموعة جزئية من زمرة بحيث تكون تخالفيه التماثل (anti-symmetric) ولكنها تكون كلية (total) في الزمرة نفسها. مما يميزها على ما يسمى بالمخروط الموجب (positive cone) والذي يحدد نوع الترتيب على زمرة مرتبة.

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ومن خلال دراسة أوراق بحثية ومراجع تتعلق بتلك المفاهيم حاولنا ان نبرز اهم خواص النطاقات غير السالبة والخواص المناظرة في ترتيب الزمر وبالطبع الخواص المناظرة لها في المخروط الموجب وان نبرز علاقة الترتيب التي تحددها مجموعة جزئية من زمرة تتمتع بخواص معينة وما يناظر تلك الخواص من صفات علاقة الترتيب من الانعكاسية والانتقالية وتخالفيه التماثل والكلية وهذا يتيح مقارنة أيا من تلك الصفات بما يقابلها من خواص المخروط الموجب او النطاقات الغير السالبة ولكي نتمكن من مقارنة النطاقات غير السالبة والمخروطات الموجبة يكون من المهم ان نناقش النطاقات غير السالبة للزمر الضربية او تحت عملية الضرب (multiplicative groups) وتناولنا أيضا في هذه الورقة بعض المفاهيم الأساسية كالعلاقة شبة المرتبة (preorder relation or quasiorder relation) وعلاقات الترتيب على المجموعات والزمر وعلاقات التكافؤ وتعريف الزمرة المرتبة جزئيا (partially ordered group) والمجموعة المحدبة (convex set) والمجموعة الموجهة (directed set) والشبكة (lattice).

1. Some definitions on groups and ordered sets:

Definition 1.1

[**11**]. A partial order on a nonempty set S is a relation ρ on S such that the following axioms are satisfied:

ρ is reflexive

$x\rho x \; for \; all \; x \in S.$

ρ is anti-symmetric

$if \; a, b \in S \; with \; a\rho b \; and \; b\rho a, then \; a = b.$

ρ is transitive

$if \; a, b, c \in S, \; with \; a\rho b \; and \; b\rho c, then \; a\rho c$.

Definition 1.2.

a partially ordered set is a set S together with a partial order relation ρ on S.

Definition 1.3.

a preorder relation on a nonempty subset A is a relation ρ on A such that the following axioms are satisfied:

The relation ρ is reflexive $x\rho x \; for \; all \; x \in S$.

ρ is transitive. $if \; a, b, c \in S, \; with \; a\rho b \; and \; b\rho c, then \; a\rho c$.

a preorder relation is also called quasiorder relation, cf.Birkhff [1]

a preorder relation induces an equivalence relation ($\sim$) on A.

Definition 1.4.

[**9**]: A semigroup is a nonempty set G together with an associative binary operation in G . A semigroup G is said to be abelian or commutative, if its binary operation is commutative this means $ab = ba$ for all $a, b \in G$.

Definition 1.5.

[**6**]: the center of a group G is defined by:

$z(G) = \{z \in G: zg = gz \ \ for \ all \ g \in G\}$

Definition 1.6.

[**5**]**:** let G be a group. G is called a tortion group. if every element of G is of finite order. While G is torsion-free, if all elements, except for the neutral element $\{e\}$, are of infinite order

From Sza'z ⌊8⌋, we have the following definitions

Definition 1.3.4.:

let X be a group. for any A, B ⊆ X, we defined

$$A^{-1} = \{x^{-1}: x \in A\} and \ A \ B = \ \{x \ y: x \ \in A, y \in B\}$$

Definition 1.7.

let G be a group and A is subset of the group G, A is called symmetric if $A^{-1} \subseteq A$

Definition 1.8.

let G be a group and A is a subset of the group G. A is called antisymmetric if A ∩ $A^{-1} \subseteq \ \{e\}$.

Definition 1.9.

let A be a subset of a group G, and B is a subset of A.B is called total in A, if A =B∪ B^{-1}.

Definition 1.10.

[7] a directed group is a partially ordered group which satisfies one, and hence all of the following statements

1. G is upper directed set, i,e. for any $a, b \in G$, there exists an upper bound of $\{a, b\}, U(a, b) \neq \phi$.
2. G is lower directed set, i,e. for any $a, b \in G$, there exists a lower bound of $\{a, b\}, L(a, b) \neq \phi$.
3. G is directed set, i,e. for any G is upper directed and lower directed of

$U(a, b) \neq \phi, L(a, b) \neq \phi.$

Definition 1.11.

A lattice – ordered groups which is a lattice under its order.

2. Some properties of Nonnegativity domains:

Definition 2.1.

[7] a total and antisymmetric subset D of a subset A of a group X is called a nonnegativity domain of A .

This means D is a nonnegativity domain, if $D \ \cap \ D^{-1} \ \subseteq \ \{e\}$ and $D \cup D^{-1} = A$

Definition 2.2.

A nonnegativity domain D of A is called multiplicative, if

$DD \subseteq \ D.$

Definition 2.3.

A nonnegativity domain D of A is called normal, if $Dx \subseteq xD$, for all $x \in A$.

Example 2.4.

$R_+ = R_+ \cup \{0\}$ and $R_+ = (0, +\infty)$ are additive non-negative domains of the additive group R of all real numbers and the subset $A = R - \{0\}$, respectively.

Example 2.5.

Consider the group $R - \{0\}$ with the usual multiplication.

Let $D_1 = (0,1)$ and $D_2 = (-1, 0)$. Then D_1 is a nonnegativity domain of $(0, \infty)$

D_2 is a nonnegativity domain of $\{(-\infty, 0) - 1\}$

$D_1 \cup D_2$ is a nonnegativity domain of $R - \{0, -1\}$

D_1 , $D_1 \cup D_2$ are multiplicative, but D_2 is not

Example 2.6.

Let G = $\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in R, ad - cb \neq 0\right\}$

Then D = $\left\{\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} : 1 < u, v < \infty\right\}$ is a nonnegativity domain of

A = $\left\{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x, y \in (0,1) \; or \; x, y \in (1, \infty)\right\}$

Proof- Since $D^{-1} = \left\{\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} : 0 < u, v < 1\right\}$, then

$D \cap D^{-1} = \emptyset \subseteq \{e\} = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ and $D \cup D^{-1} = A$

Theorem 2.7. If D is a nonnegativity domain of a subset A of a group G then $e \in D$ if and only if $e \in A$. Therefore if $e \in A$, then $D \cap D^{-1} = \{e\}$

Proof- Since $D \subseteq A$, if $e \in D$, then $e \in A$. Conversely, if $e \in A$, then $e \in D \cup D^{-1}$.

This means $e \in D$ or $e \in D^{-1}$. However, $e \in D$ implies $e = e^{-1} \in D^{-1}$ and

$e \in D^{-1}$ implies that $e^{-1} = e \in (D^{-1})^{-1} = D$. Therefore $\{e\} \subseteq D \cap D^{-1}$. By antisymmetry of D, we have $D \cap D^{-1} \subseteq \{e\}$. Consequently $D \cap D^{-1} = \{e\}$.

Theorem 2.8 Let D be a nonnegativity domain of a subset D of a group G

If $e \in D$ and D is multiplicative, then $DD = D$

Proof- If $e \in D$, then we also have $D = eD \subseteq DD$. If D is in addition multiplicative, then $DD \subseteq D$ Therefore $DD = D$

Theorem 2.9 If D is a nonnegativity domain of a subset A of a group G, then A is a symmetric subset of G.

Proof- To prove A is symmetric, let $x \in A^{-1}$, then $x^{-1} \in A$. Since $A = D \cup D^{-1}$, then $x^{-1} \in D \cup D^{-1}$, i.e $x^{-1} \in D$ or $x^{-1} \in D^{-1}$. If $x^{-1} \in D$, then $x \in D^{-1}$, consequently $x \in A$. If $x^{-1} \in D^{-1}$, then $x \in D$, therefore $x \in A$. Thus $A^{-1} \subseteq A$. Then A is a symmetric subset of G .

Theorem 2.10. Let D be a nonnegativity domain of a subset A of a group G , if D is normal in A , then $Dx = xD$ and $Ax = xA$ for all $x \in A$

Proof- Since D is normal in A , then $Dx \subseteq xD$, for all $x \in A$. From

Theorem 2.11. A is symmetric, consequently, $x^{-1} \in A$ for all $x \in A$.

Therefore $Dx^{-1} \subseteq x^{-1}D$. By using the cancellation laws, we get $x(Dx^{-1})x \subseteq x(x^{-1}D)x$. Thus $xD \subseteq Dx$. Therefore $xD = Dx$ for all $x \in A$. Consequently $x^{-1}D = Dx^{-1}$ for all $x \in A$. Therefore, we also have $D^{-1}x = \{y^{-1}x: y \in D\} = \{(x^{-1}y)^{-1}: y \in D\} = (x^{-1}D)^{-1} = (Dx^{-1})^{-1} = xD^{-1}$.

Thus $xA = x(D \cup D^{-1}) = xD \cup xD^{-1} = Dx \cup xD^{-1}x = (D \cup D^{-1})x = Ax$.

Lemma 2.12. Let A, B be subsets of the group G. We have the following statements:

$i. (A \cap B)^{-1} = A^{-1} \cap B^{-1}$

$ii. (A \cup B)^{-1} = A^{-1} \cup B^{-1}$

$iii. (A^{-1})^{-1} = A$

Proof . $(A \cap B)^{-1} = \{x^{-1}: x \in A \cap B\} = \{x^{-1}: x \in A \text{ and } x \in B\}$
$= \{x^{-1}: x \in A\} \cap \{x^{-1}: x \in B\} = A^{-1} \cap B^{-1}$

$ii. (A \cup B)^{-1} = \{x^{-1}: x \in A \cup B\} = \{x^{-1}: x \in A \text{ or } x \in B\}$
$= \{x^{-1}: x \in A\} \cup \{x^{-1}: x \in B\} = A^{-1} \cup B^{-1}$

$iii. (A^{-1})^{-1} = \{x^{-1}: x \in A^{-1}\} = \{x^{-1}: x^{-1} \in A\} = \{y: y \in A\} = A$.

Theorem 2.13. If D is a nonnegativity domain of a subset A of a group G, B is a symmetric subset of A and $E = D \cap B$, *then* E *is a nonnegativity domain of* B.

Proof *We prove* $E \cup E^{-1} = B$ *and* $E \cap E^{-1} = \{e\}$.

Let $x \in E \cup E^{-1}$ *this means* $x \in E$ *or* $x \in E^{-1}$. If $x \in E$, since $E \subseteq B$. This means

$x \in B$……. (1)

If $x \in E^{-1}$, then $x \in B^{-1}$. Since B is symmetric, then $x \in B$…… (2)

From (1) and (2) we get $E \cup E^{-1} \subseteq B$.

If $x \in B$, then $x \in E$ or $x \notin E$.

If $x \in E$, then $x \in E \cup E^{-1}$……… (1)

If $x \notin E$, since $E = D \cap B$ and $D \cap D^{-1} \subseteq \{e\}$, then $x \in D^{-1}$

Consequently $x^{-1} \in D$. Since $x \in B$, then $x^{-1} \in B^{-1}$. Since B is symmetric, then $x^{-1} \in B$. Therefore $x^{-1} \in D \cap B$ i.e $x^{-1} \in E$. Consequently $x \in E^{-1}$. Hence from (1) and (2) we get $B \subseteq E \cup E^{-1}$.

Therefore $B = E \cup E^{-1}$. $E \cap E^{-1} \subseteq D \cap D^{-1} \subseteq \{e\}$

Example 2.14. By theorem 3.1.9, we have $Z_{\oplus} = Z \cap R_{\oplus}$ and $Q_{\oplus} = Q \cap R_{\oplus}$ are additive non-negativity domains of the additive groups Z and Q of all integer and rational numbers, respectively.

Theorem 2.15. If D is a non-negativity domain of a subset A of a group X, then $E = D^{-1}$ is also a non-negativity domain of A. Moreover, if D is a multiplicative (normal in A), then E is also multiplicative (normal in A).

Proof. Since D is a non-negativity domain, then $D \cap D^{-1} \subseteq \{e\}$ and $D \cup D^{-1} = A$

Since $E = D^{-1}$, then $E^{-1} = D$. This means $E^{-1} \cap E \subseteq e$, $E^{-1} \cup E = A$. Thus, E is a non-negativity domain of A

To prove E is a multiplicative, let $x \in D^{-1}D^{-1}$, then $x = d_1^{-1}d_2^{-1}$ for some $d_1, d_2 \in D$. Thus $x^{-1} = d_2 d_1 \in DD \subseteq D$, then $x^{-1} \in D$, consequently $x \in D^{-1}$. So $D^{-1}D^{-1} \subseteq D^{-1}$. This means E $e \in D$ is a multiplicative.

To prove E is a normal in A, we have $xD \subseteq Dx$, for all $x \in A$.

Show $yD^{-1} \subseteq D^{-1}y$ for all $y \in A$. Let $x \in yD^{-1}$, we have $x = yd^{-1}$, for some $d \in D$. Now multiply by D we get $y = xd$, then $y \in xD$, since D is a normal, then $y \in Dx$. This means there exists $d^* \in D$ such that $y = d^*x$, consequently

$x = (d^*)^{-1}y$. This means $x \in D^{-1}y$. Thus $yD^{-1} \subseteq D^{-1}y$.

Theorem 2.16. If D and E are non-negativity domains, of a subset A of a group X, with $e \in A$, such that $D \subset E$, then $D = E$

Proof. If $x \in E$, since $E \subset A = D \cup D^{-1}$. We have either $x \in D$ or $x \in D^{-1}$.

If $x \in D^{-1}$, then $x^{-1} \in D$ since $D \subset E$, then $x^{-1} \in E$, Consequently $x \in E^{-1}$

Therefore $x \in E \cap E^{-1} \subseteq \{e\}$. Thus $x = e$, then $x \in D$.

Therefore $E \subset D$ is also true. Consequently $E = D$.

Theorem 2.17. If A and B are symmetric subsets of the group X and Y, respectively E is a non-negativity domain of B and f is an odd function of A into B such that $e \notin f(A/\{e\})$, then $D = f^{-1}(E)$ is a non-negativity domain of A

Proof Let $x \in f^{-1}(E^{-1})$ then $f(x) \in E^{-1}$. This means $(f(x))^{-1} \in E$. Since f is odd, then $f(x^{-1}) \in E$. Consequently $x^{-1} \in f^{-1}(E)$, this means $x \in (f^{-1}(E))^{-1}$. Conversely, if $x \in (f^{-1}(E))^{-1}$, then $x^{-1} \in f^{-1}(E)$. This means $f(x^{-1}) \in E$. Consequently $(f(x))^{-1} \in E$, i.e $f(x) \in E^{-1}$. Thus $x \in f^{-1}(E^{-1})$

Since $e \notin f(A\setminus\{e\})$, we can see that $x \in f^{-1}(\{e\})$, implies $f(x) = e$, then

Theorem 2.18.
$if\ A\ and\ B\ are\ symmetric\ subsets\ of\ the\ groups\ X\ and\ Y,$
$respectively\ E\ is\ a\ nonnegativity\ domain\ of\ B\ and\ f\ is\ an\ odd\ function\ of\ A$
$into\ B\ such\ that\ e \notin f(A\{e\}), then\ D$
$= f^{-1}(E)\ is\ a\ nonnegativity\ domain\ of\ A$

$\boldsymbol{Proof}.\ Let\ x \in f^{-1}(E^{-1})\ then\ f(x) \in E^{-1}. This\ means\ \left(f(x)\right)^{-1}$
$\in E\ .Since\ f$

$is\ odd, f(x^{-1}) \in E\ .Consequently\ x^{-1} \in f^{-1}(E), this\ means\ x \in (f^{-1}(E))^{-1}.$

$Conversely\ , if\ x \in (f^{-1}(E))^{-1}\ , then\ x^{-1} \in f^{-1}(E). This\ means\ f(x^{-1}) \in E\ .$

$Consequently\ (f(x))^{-1} \in E\ , i\ .e\ f(x) \in E^{-1}. Thus\ x \in f^{-1}(E^{-1})$

$Since\ e \notin f(A\backslash\{e\})\ , we\ can\ see\ that\ x \in f^{-1}(\{e\}), implies\ f(x) = e\ , then\ x \notin A\{e\}\ , this\ means\ x = e\ .Therefore\ , (f^{-1}(E))^{-1}, and\ f^{-1}(\{e\})\ \subseteq \{e\}.$

$Since\ E\ is\ a\ non-negativity\ domain\ of\ B\ .We\ have$

$$A = f^{-1}(B) = f^{-1}(E \cup E^{-1}) = f^{-1}(E) \cup f^{-1}(E^{-1})$$
$$= f^{-1}(E) \cup (f^{-1}(E))^{-1}$$
$$= D \cup D^{-1}\ .$$

$Now\ D \cap D^{-1} = f^{-1}(E) \cap \left(f^{-1}(E)\right)^{-1} = f^{-1}(E) \cap f^{-1}(E^{-1}) = f^{-1}(E \cap E^{-1}) = f^{-1}(e) \subseteq \{e\}.\ Therefore, D\ is\ non-negativity\ domain\ of\ A.$

We can also state:

Corollary 2.19

$If\ A\ and\ B\ are\ symmetric\ subsets\ of\ the\ groups\ X\ and\ Y\ , respectively, D\ is\ a\ non-negativity\ domain\ of\ A\ and\ f\ is\ an\ odd\ injective\ function\ of\ A\ onto\ B\ such\ that\ f(e) = e, if\ e \in A, then\ E = f(D)\ is\ a\ non-negativity\ domain\ of\ B\ .$

$Proof\ .Since\ D\ is\ a\ non-negativity\ domain\ of\ A\ and\ f\ is\ odd.\ We\ have$

$$B = f(A) = f(D \cup D^{-1}) = f(D) \cup f(D^{-1})\ f(D) \cup \left(f(D)\right)^{-1} = E \cup E^{-1}.$$

Now, $E \cap E^{-1} = f(D) \cap \left(f(D)\right)^{-1} = f(D) \cap \left(f(D^{-1})\right) \subseteq f(D \cap D^{-1}) \subseteq f(e) = e$

Theorem 2.20. $if\ D\ is\ a\ non-negativity\ domain\ of\ a\ subset\ A\ of\ a\ group\ X\ and\ f\ is\ an\ injective\ multiplicative\ function\ of\ X\ into\ a\ group\ Y\ , then\ E = f(D) is\ a\ non-negativity\ domain\ of\ B = f(A). Moreever, if\ D\ is\ a\ multiplicative\ (normal\ in\ A), then\ Eis\ a\ multiplicative\ (normal\ in\ B)$

$Proof. Since\ f\ is\ a\ multiplicative, this\ means$

$$f(e)f(e) = f(e), then\ f(e) = e$$

$We\ also\ have\ f(x)f(x^{-1}) = f(e) = e.\ This\ means\ f(x^{-1}) = \left(f(x)\right)^{-1}$

For all x

From theorem 3.1.6 A is symmetric, and since $B^{-1} = (f(A))^{-1} = f(A^{-1}) = f(A) = B$. Then B is symmetric by Corollary 3.1.12 we get, E is a non-negativity domain of B.

To prove the remaining assertions, then

$EE = f(D)f(D) = f(DD) \subset f(D) = E$. Consequently, E is also multiplicative. If D is normal in A, then

$E = f(x) = f(D)f(x) = f(Dx) \subset f(xD) = f(x)f(D) = f(x)E.$ For all $x \in A$.

Since $B = f(A)$, we also have $Ey = yE$ for all $y \in B$. Therefore E is normal.

3.the positive cone.

After we studied the nonnegativity domain on multiplication groups to make it more comparably with positive cones, in this section we will talk about some properties of orderability on sets and groups which is related with the concept of positive cone.

Definition 3.1. [7]: a partially ordered group is a set such that

1. G is a partially ordered set under a relation $\leq$.
2. G is a group.
3. $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$ for all $a, b, c \in G$.

Remark. statement 3 is called the monotony law.

Theorem 3.2 let G be a partially ordered set. If G is a group, then the following statements are equivalent:

1. $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$ for all $a, b, c \in G$.
2. $a \leq b$ implies $cad \leq cbd$ and $ca \leq cb$ for all $a, b, c, d \in G$.
3. $a < b$ implies $ca < cb$ and $ac < bc$ for all $a, b, c \in G$.
4. $a \leq b$ and $c \leq d$ imply $ac \leq bd$ for all $a, b, c, d \in G$.
5. $a < b$ and $c < d$ imply $ac < bd$ for all $a, b, c, d \in G$.

If G is a partially ordered group under a partial ordered relation $\leq$, then G together with the dual of $\leq$, then G together with the dual of $\leq$ (i.e. $\geq$) is a partially ordered group.

We may write the following alternative and equivalent definition:

Definition 3.3. a partially ordered group (p.o.group) is a set G satisfying the following axioms:

1.There exists a partial ordered relation $\leq$ on the set G.

2. G is group.

3.One (and, hence, all) of the statements of Theorem3.2. is satisfied.

Mutsushita [12] and Zaiciceva [14] considered the case when only the half of the monotony law, $a \leq b$ implies $ca \leq cb$ for all $a, b, c \in G$, in definition 2.1 is assumed (see also Conrad [4] and Cohn [3]).

A somewhat more general notion the partially ordered group has been studied by Britton and shepherd [2] under the name “almost ordered group”

If condition 1 in the definition 2.1.6 is weakened, then we have.

Theorem 3.4[7] let $\leq$ be a preorder relation on a group G such that $a \leq b$ implies $ca \leq cb$ and $ac \leq bc$ for all $a, b, c \in G$. If a relation, Denoted by $\sim$, is defined on G by

$x \sim y\ \ if\ and\ only\ if\ x \le y\ and\ y \le x$,

Then

(i) $x \sim y\ \ if\ and\ only\ if\ xy^{-1} \sim e(if\ and\ only\ if\ y^{-1}x \sim e)$.

(ii) $x_1 \sim y_1\ and\ x_2 \sim y_2\ imply\ x_1x_2 \sim y_1y_2$.

(iii) $The\ equivalence\ class\ N = [e] = \{x \in G: x \sim e\}$ $is\ a\ normal\ subgroup\ of\ G$.

(iv) ${}^{G}/_{N} = \{[a]: a \in G\} = \{aN: a \in G\}, in\ fact, aN = [a]$ $for\ all\ a \in G, where\ [a]\ is\ the\ equivalent\ class\ of\ a.$

(v) ${}^{G}/_{N}$ $is\ a\ partially\ ordered\ group\ under\ the\ relation$ $induced\ by\ \ \le$.

Proof.

(i) Let $x \sim y$ this means $x \le y$ and $y \le x$. Multiply by y^{-1}. We get $xy^{-1} \le e$ and $e \le xy^{-1}$. Thus $xy^{-1} \sim e$.
$conversely\ \ xy^{-1} \sim e, then\ \ xy^{-1} \le e$ and $e \le xy^{-1}$.
Multiply by $y, we\ get\ x \le y$ and $y \le x$,i.e. $x \sim y$.

(ii) Let $x_1 \sim y_1$ and $x_2 \sim y_2$. Since $x_1 \sim y_1$, then $x_1 \le y_1$ and $y_1 \le x_1$. $Since\ x_2 \sim y_2\ then\ x_2 \le y_2\ and\ y_2 \le x_2$.
$multiply\ x_1 \le y_1\ by\ x_2, then\ x_1 x_2 \le y_1 x_2$. $multiply$ $x_2 \le y_2\ by\ y_1, then\ y_1x_2 \le y_1 y_2. Now\ x_1x_2 \le y_1 x_2$ $and\ y_1x_2 \le y_1 y_2\ imply\ x_1x_2 \le y_1 y_2$.
$Similarly, it\ can\ be\ shown\ that\ x_1x_2 \ge y_1 y_2$.
$Therefore\ x_1x_2 \le y_1 y_2\ and\ x_1x_2 \ge y_1 y_2$.
$This\ means\ x_1x_2 \sim y_1 y_2$.

(iii) We shall prove that N is a subgroup. Let $x, y \in N$, then $x \sim e$ and $y \sim e$. Then $x \le e, e \le x, y \le e\ and\ e \le y$.

Then $y^{-1} \le e, e \le y^{-1}$. Then $xy^{-1} \le x\ and\ x \le xy^{-1}$. This together with $x \le e, e \le x$, by the transitive law, imply that $xy^{-1} \le e, e \le xy^{-1}. Thus\ xy^{-1} \sim e\ and\ xy^{-1} \in N\ forx, y \in N. Therfore\ N\ is\ a\ subgroup.$

(iv) We will show that $\{aN: a \in G\} = \{[a]: a \in G\}$. In fact, we will Show that $aN = [a]$ for all $a \in G$. Let $y \in aN$. Then $y = ax\ for\ some\ x \in N$. Since $x \le e$ and $x \ge e$, then $y = ax \le ae$ and $y = ax \ge ae$. Then $y \sim a$ and $y \in [a]$.
$aN \subseteq [a]$.
$on\ the\ hand, if\ y \in [a], then\ y \ge a\ and\ y \le a. Then\ a^{-1}y \ge e$ $and\ a^{-1}y \le e. Then\ x = a^{-1}y \in N\ and\ y = ax. Thus\ y \in aN.$
So $[a] \subset aN$. $consequently\ aN = [a]$.

(v) Defined a relation on ${}^{G}/_{N}$ by:

$$[a] \le [b]\ if\ and\ only\ if\ a \le b.$$

First, we will show that the relation is well-defined, i.e. If $[a] \leq [b]$ and $a' \in [a], b' \in [b]$, we shall show that $[a'] \leq [b']$. We have

Definition 3.5. [7]: let $G\ be\ a\ partialy\ ordered\ group\ \ and\ let\ x\epsilon G.then$

I. $x\ is\ called\ positive\ or\ inegral\ \ if\ x \geq e.$

II. $x\ is\ called\ strictaly\ positive\ or\ strictly\ integral\ ,if\ x > e\ .$

III. $x\ is\ called\ negative\ if\ x \leq e.$

Definition 3.6. [7]: let $G\ be\ a\ partialy\ ordered\ group,\ the\ set$ $G^{+} = U(e) = \{x \in G: x \geq e\}$ of all positive (integral) elements of G is denoted $by\ p(G)\ or, simply, P\ and\ is\ called\ the\ positive\ cone(or\ the\ integral)of\ G$

Lemma 3.7. let $G\ be\ a\ partially$ ordered group then the following statement are equivalent.

$$(i)\ a \leq b.$$
$$(ii) ba^{-1} \in P.$$
$$a^{-1}b \in P.$$
$$Where\ P = \{x \in G: x \geq e\}\ is\ the\ positive\ cone\ of\ G.$$

Proof. Suppose statement 1 holds. This means $a \leq b$. Multiply by $a^{-1}\ from\ the\ right, we\ get$

$$aa^{-1} \leq ba^{-1}, i.e\ ba^{-1} \geq e. This\ means\ ba^{-1} \in P.$$

Suppose statement 2 holds. This means $ba^{-1} \geq e, multiply\ by\ a\ from\ the\ right, we\ get\ b \geq a.$

This means $a^{-1}b \in P$.

Suppose statement 3 holds. This means $a^{-1}b \geq e, multiply\ by\ a\ from\ the\ left\ we\ get\ b \geq a$

Theorem 3.8 [13]

$Let\ G\ be\ a\ partially\ ordered\ group.\ Then\ the\ positive\ cone\ P\ of\ G\ \ satisfies\ the\ following\ properties$:

$(i)PP \subseteq P.$

$(ii)\ P \cap P^{-1} = \{e\}.$

$(iii)\ x^{-1}Px \subseteq P\ (or, equivalently\ xPx^{-1} \subseteq P) for\ all\ x \in G.$

Proof. $To\ prove\ (i).\ let\ x, y\ \in p\ then\ x \geq e\ and\ y \geq e\ multiply\ the\ first\ inequality\ by\ y, we\ get\ xy \geq y\ and\ y \geq e. The\ transitivity\ of\ \leq\ implies\ that\ xy \geq e. Hence\ xy \in P\ and\ PP \subseteq P.$

$To\ prove\ (ii). let\ x, y\ \in p\ then\ x \geq e\ and\ e = e^{-1} \in P^{-1}. Then\ \{e\}\ \subseteq P \cap P^{-1}. Also\ P \cap P^{-1} \subseteq \{e\}\ which\ follows\ from\ the\ fact\ that\ x \in P \cap P^{-1}, then\ x \in P\ and\ x \in P^{-1}. Then\ x \in P\ and\ x^{-1} \in P, i.e\ x \geq e\ and\ \ x^{-1} \geq e.\ Then\ x \geq e\ and, on\ multiplying\ x^{-1} \geq e$

$by\ x, e \geq x. The\ antisymmetry\ of\ \leq implies\ x = e,\quad i.e$
$P \cap P^{-1} \subseteq \{e\}.\ Thus\ P \cap P^{-1} = \{e\}.$

To prove(iii) we first note that if $y \in xPx^{-1}, then\ x^{-1}yx \in P.$
$This\ means\ x^{-1}yx \geq e\ muliply\ by\ x\ from\ the\ right\ and\ x^{-1} from\ the\ left,$
$we\ get\ y \leq e. This\ means\ y \in P. Consequently\ x^{-1}Px \subseteq P.$

Example. Let G be the additive group of complex numbers and define
$x + iy \leq u + iv\ if\ and\ only\ if\ x \leq u\ and\ y \leq v. Then\ the\ positive\ cone\ P = \{x + iy: x \geq 0, y \geq 0\}.\ One\ may\ certianly\ say\ that$

(i) $P + P \subseteq P.$

(ii) $P \cap P^{-1} = \{0\}, and$

(iii) $x + P - x \subseteq P(or, equivalently\ -x + P + x \subseteq P)\ For\ all\ x \in G. In\ fact, \langle\subseteq\rangle\ can\ be\ replaced\ by \langle=\rangle\ in\ (i)\ and\ (iii).$

As a corollary of the above theorem

Theorem
3.9. $The\ positive\ cone\ of\ a\ partially\ ordered\ group\ G\ has\ the\ following\ properties$

$(i)\ P\ is\ a\ semigroup$

$(ii) P\ is\ a\ normal\ in\ G$

Corollary 3.10.
$if\ P\ is\ the\ positive\ cone\ of\ a\ partially\ ordered\ group\ G\ then$:

$(i)\ PP = P.$

$(ii)\ xPx^{-1} = P\ for\ all\ x \in G.$

$Proof. (i)\ From\ Theorem\ 2.4.4\ PP \subseteq P. If\ x \in P\ then\ x = ex \in PP. This\ means\ P \subseteq PP. This\ means\ P \subseteq PP.$

$(ii)\ From\ Theorem\ 2.4.4\ xPx^{-1} \subseteq P. If\ p \in P, then\ p \geq e. Multiply\ by\ x^{-1}\ and\ x, we\ get\ x^{-1}px \geq e. This\ means\ x^{-1}px \in P, then\ x\ (x^{-1}px)x^{-1} \in xP\ x^{-1}. Therefore\ P \subseteq xP\ x^{-1}. Thus\ P = xP\ x^{-1}.$

Theorem 3.11. let G be a group and p a nonempty subset of G and defined a relation $\leq$ on G by a $\leq$ b if and only if $a\, b^{-1} \in p$. Then:
(i) $\leq$ is reflexive if and only if $e \in p$.
(ii) $\leq$ is antisymmetric if and only if $p \cap p^{-1} \subseteq \{e\}$.

(iii)$\leq$ is a transitive if and only if $pp \subseteq p$

(iv) the monotony law i.e. the condition
$(a \leq b\ implies\ xay \leq xby\ for\ all\ x, y \in G)\ holds\ if and\ only\ if\ xpx^{-1} \subseteq p, for\ all\ x \in g$

Proof.

(i) Suppose $\leq$ is reflex then $a \leq a$ for all $a \in G$. This means $aa^{-1} \in P$e. e $\in P$.

Conversely if e $\in P$, then for any $a \in G$, $e = aa^{-1} \in P$. Therefore $a \leq a$, and thus $\leq$ is reflexive.

$b^{-1} \leq a, this\ means\ a(b^{-1})^{-1} \in P, consequently\ ab \in P. Since\ a, b, \in P\ are\ arbitrary, then\ PP \subseteq P. Conversely, suppose\ PP \subseteq P. If\ a \leq b, b \leq c, then\ ba^{-1} \in P, cb^{-1} \in P, and\ so\ cb^{-1}ba^{-1} = ca^{-1} \in PP \subseteq P. This\ means\ a \leq c\ and\ so\ the\ relation\ is\ transitive.$

(iv) The condition (a$\leq$ b implies $xay \leq xby$) holds if and only if the condition $ba^{-1} \in P$ implies (xby) $(xay)^{-1} \in P$ holds, This is equivalent to $ba^{-1} \in P$ implies x$(ba^{-1})x^{-1} \in P$. This condition is clearly true if $xPx^{-1} \subseteq P$. Conversely id the monotony law holds, i.e the condition $ba^{-1} \in P$ implies x$(ba^{-1})x^{-1} \in P$ holds, then, since every element b of P is of the form $ba^{-1} \in P$ where $a = e$, we have $x(ba^{-1})x^{-1} = x(b)x^{-1} \in P$ which means $xPx^{-1} \subseteq P$.

In the above theorem, the condition $ba^{-1} \in P$ can be replaced by $a^{-1}b \in P$ to obtain the following :

Theorem3.12.

$Let\ G\ be\ a\ group\ and\ P\ a\ nonempty\ subset\ of\ G\ and\ defined\ a\ relation\ \leq on\ G\ by\ a \leq b\ if\ and\ only\ if\ a^{-1}b \in P, then$

$(i)\ \leq is\ reflexive\ if\ and\ only\ if\ e \in P.$

$(ii)\ \leq is\ antisymmetric\ if\ and\ only\ if\ P \cap P^{-1} \subseteq \{e\}.$

$(iii)\ \leq is\ transitive\ if\ and\ only\ if\ PP \subseteq P.$

$(iv)\ The\ monotony\ law(i.e\ The\ condition\ a \leq b\ implies\ xay \leq xby)\ holds\ if\ and\ only\ if\ xPx^{-1} \subseteq P.$

Proof. Similar to the proof of the above theorem.

Example. Let G=R/{0} and $P = [1, \infty)$, then P satisfies the following statements:

4. The Existence of Nonnegativity Domains:

Definition 4.1. $A\ subset\ of\ A\ of\ a\ group\ X\ is\ called\ n-cancellable\ for\ some\ n \in N\ if\ x^n = e\ implies\ x = e\ for\ all\ x \in A.\ Equivalently, if\ x \in A - \{e\}\ implies\ x^n \neq e.$

Remark. If we defined $X_n = \{x \in X; (x)^n = e\},\ then\ X\ is\ n-cancellable\ if\ and\ only\ if\ X_n = \{e\}.\ This\ means\ \{e\} \cup (X \backslash X_n) is\ the\ largest\ n-cancellable\ subset\ of\ X.$

Example. $if\ A = \{1, -1, i, -i\} is\ a\ subset\ of\ the\ multiplicative\ group\ of\ all$

$nonzero\ complex\ numbers.\ Since\ x \in A - \{1\} implies\ x^{2k+1} \neq 1, then\ A\ is$
$2k+1 - cancellable\ for\ all\ k.\ But\ A\ is\ not\ 2k - cancellable\ as\ -1 \in A - \{e\}$
$and (-1)^{2k} = 1.$

Example. $if\ B = \{1, i, -i\}\ is\ a\ subset\ of\ the\ multiplicative\ group\ of\ all$
$nonzero\ complex\ numbers. Since\ x \in B - \{1\}\ implies\ x^{4k} = 1, then\ B\ is\ not$
$$4k - cancellable$$
$But\ x^{4k+1} = x \neq 1, x^{4k+2} = x^2 = -1 \neq 1, x^{4k+3} = x^3 = -x \neq -1,$
$then\ B\ is\ 4k+1, 4k+2, 4k+3 - cancellable.$

Theorem 4.2. $if\ a\ subset\ of\ a\ group\ x\ has\ a\ non-negativity\ domain,$
$then\ A\ is\ 2 - cancellable.$

$Proof. Since\ D\ is\ a\ non-negativity\ domain\ of\ A\ and\ assume\ that\ x \in A,$
$such\ that\ x^2 = e, this\ means\ x = x^{-1}.\ Since\ A = D \cup D^{-1}\ we\ have\ either$
$x \in D\ or\ x \in D^{-1}. If\ x \in D, since\ x^{-1} = x\ we\ also\ have\ x^{-1} \in D, i.e\ x \in D^{-1},$
$Similarly, if\ x \in D^{-1}, since\ x^{-1} \in D^{-1} i.e\ x \in D. Consequently$
$x \in D \cap D^{-1} \subseteq \{e\}, and\ hence\ x = e.$

Example.
$If\ A\ is\ a\ subset\ of\ the\ multiplicative\ group\ of\ all\ nonzero\ complex\ numbers\ such\ that\ -1 \in A, then\ A\ has\ no\ non-negativity\ domain.\ It\ is\ enough\ to\ note\ only\ that\ (-1)^2 = 1, but\ -1 \neq 1.\ Thus, A\ is\ not\ 2\ cancellable$

Theorem 4.3. $If\ a\ is\ a\ symmetric\ and\ 2-cancellable\ subset\ of\ a\ group\ X$ l
$and\ B\ is\ an\ anti-symmetric\ of\ A\ then\ there\ exists\ a\ non-negativity$
$domain\ D\ of\ A\ such\ that\ B \subseteq D.$

$Proof. Let\ \mathcal{D}\ be\ the\ family\ of\ all\ anti-symmetric\ subsets\ D\ of\ A\ such\ that$

$B \subset D.$ *This means* $, B \in \mathcal{D},$ *and thus* $\mathcal{D} \neq \emptyset.$ *Moreover, since the union of a directed family of anti − symmetric subets of* X *is also anti − symmetric ,*
then $\mathcal{D}$ *is, in particular, inductive. By Zorn's Lemma, there exists maximal element* D *of* $\mathcal{D}$
since D *is an anti − symmetric subset of* A *such that* $B \subset D$ *and*
$D \cup D^{-1} \subset A \cup A^{-1} = A.$ *To show that* $A \subset D \cup D^{-1}$*, for this assume on the contrary that there exists* $x \in A$ *such that* $x \notin D$ *and* $x^{-1} \notin D.$ *Define*
$E = D \cup \{x\}.$ *Then we have* $B \subseteq E \subseteq A.$
Moreover, if $y \in E \cap E^{-1}$*, i.e,* $y \in D \cup \{x\}$ *and* $y^{-1} \in D \cup \{x\}$*, then by examining the four possible cases and using the assumptions* $x \notin D$ *and*
$x^{-1} \notin D$*, we can see that either* $y \in D \cap D^{-1}$ *or* $y^2 = e$ *can hold*
By the anti − symmetry of D *and the* 2 *− cancellability of* A*, it follows that* $y = e$ *. Therefore* $E \cap E^{-1} \subset \{e\}$*, thus* $E \in \mathcal{D}$ *. Hence by using the maximality of* $D \subset E$ *, we can infer that* $E = D$ *, which is a contradiction . Therefore, the required assertion is true .*

Theorem 4.5. *If* A *is a subset of a group* X *then the following assertions are* Equivalent:

1. A has a non-negativity domain .
2. A is symmetric and 2-cancellable .

Proof. Suppose (1) *holds by Theorem*4.2 *we get* A *is symmetric, and by Theorem* 4.3 *we get* A *is* 2 *− cancellable Conversely, since* $\emptyset$ *is an anti − symmetric subset of* A *. Suppose statement* (2) *hold . By putting* $B = \emptyset$ *in Theorem* 4.2 *we get statement* (1).

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